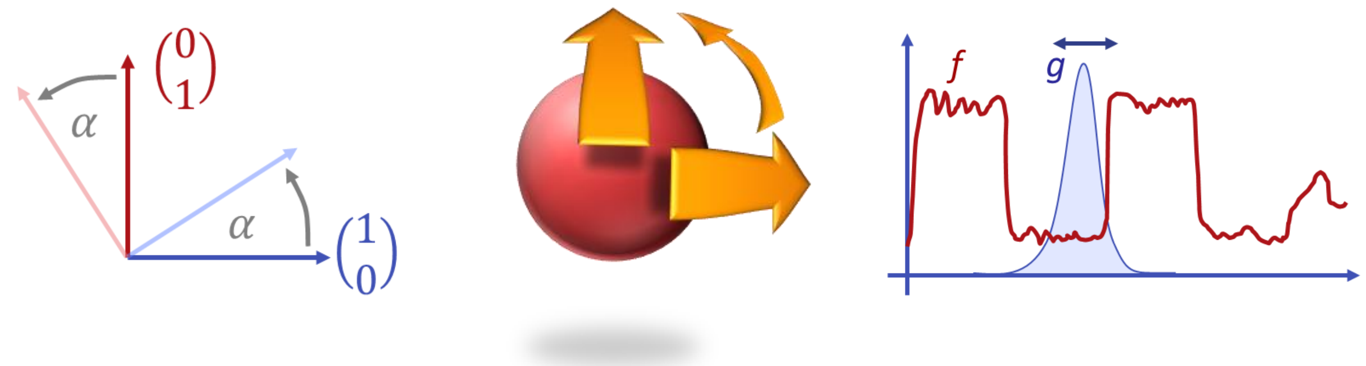
Modelling 1

SUMMER TERM 2020



LECTURE 8

(Linear) Information Loss

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Information Loss
in Linear Mappings

Linear Maps

A function

- $f: V \to W$ between vector spaces V, W

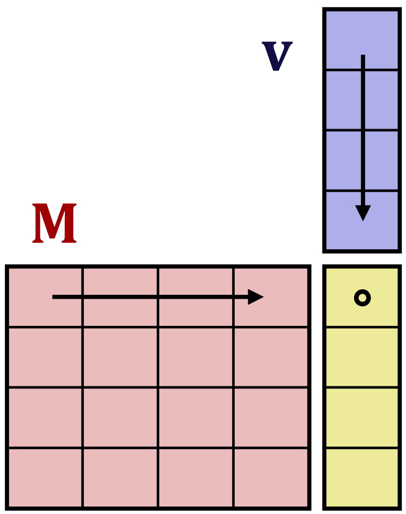
is linear if and only if:

- $\forall \mathbf{v}_1, \mathbf{v}_2 \in V$: $f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$
- $\forall \mathbf{v} \in V, \lambda \in \mathbb{R}$: $f(\lambda \mathbf{v}) = \lambda f(\mathbf{v})$

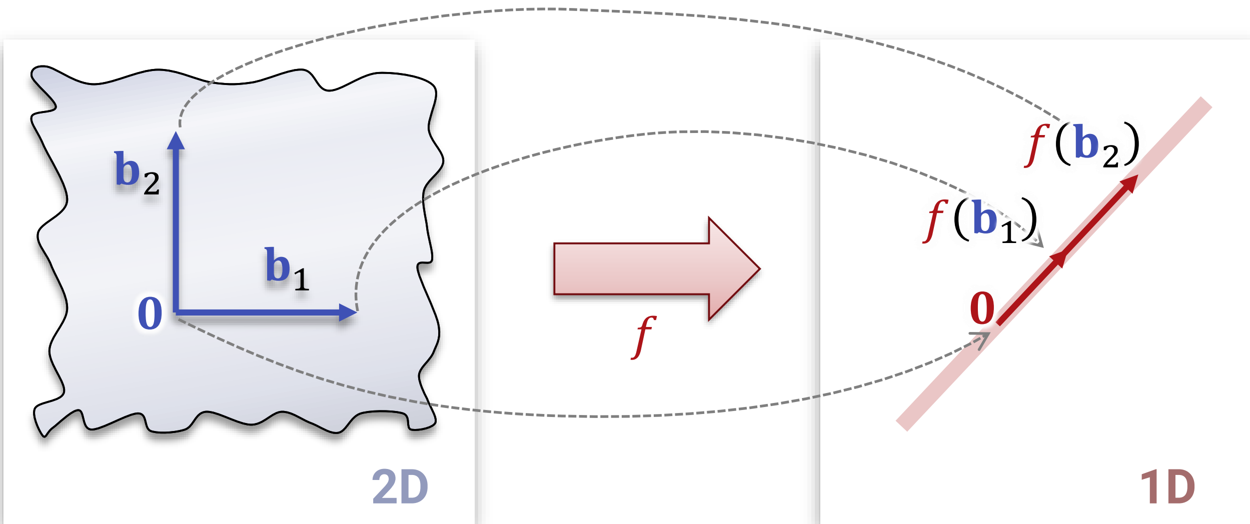
Matrix Product

All operations are matrix-matrix products:

- Matrix-Vector product:
- $f(\mathbf{x}) = \mathbf{M}_f \cdot \mathbf{x}$



Not invertible

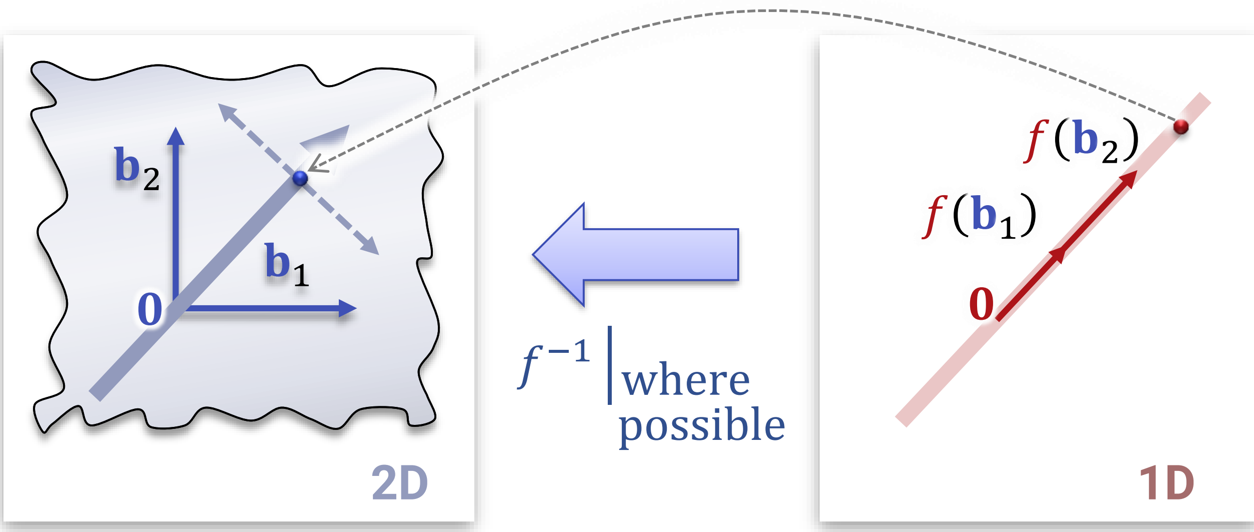


$$f(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \cdot \mathbf{x}$$

Information flow:

- After f, we can recover $b_1 + b_2$
  - Sum of inputs
- We do not know $b_1 - b_2$ anymore
  - Difference of inputs

Not invertible

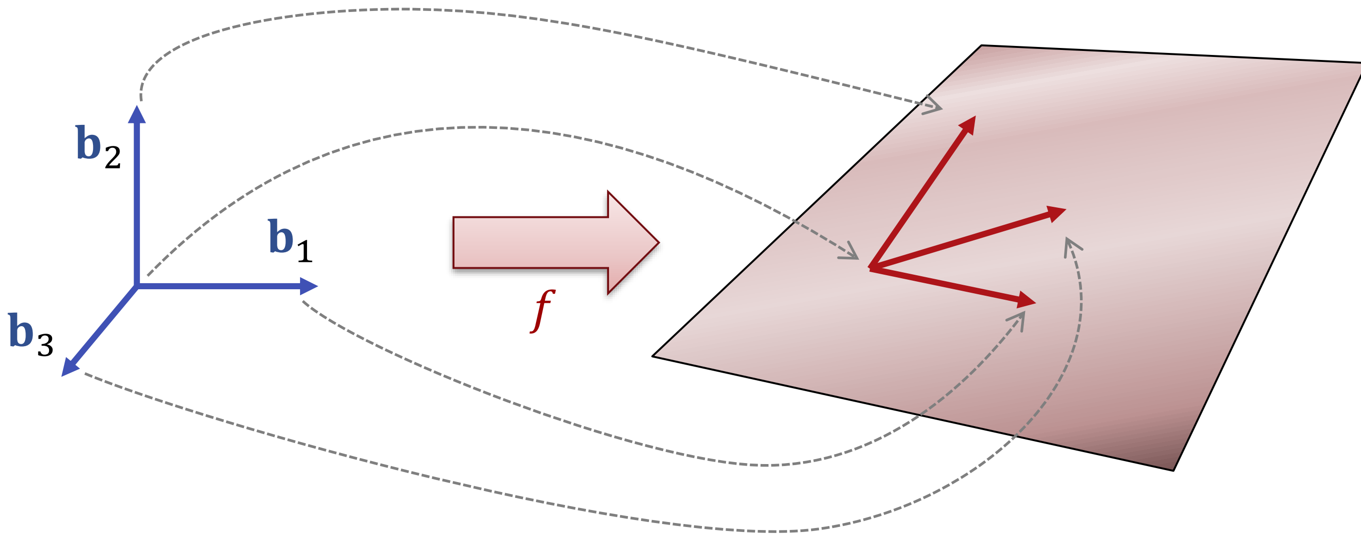


$$f(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \cdot \mathbf{x}$$

Information flow:

- After f, we can recover $b_1 + b_2$
  - Sum of inputs
- We do not know $b_1 - b_2$ anymore
  - Difference of inputs
  - Anything along that line, i.e. $(\lambda, -\lambda), \lambda \in \mathbb{R}$

Not invertible



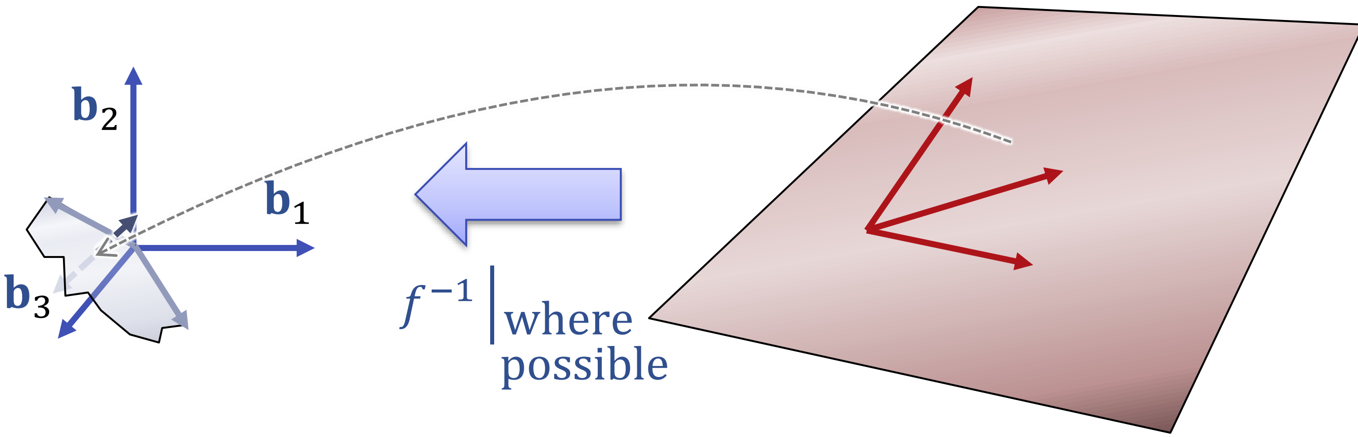
$$f(\mathbf{x}) = \begin{pmatrix} x_1 + x_3 \\ x_2 + x_3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \mathbf{x}$$

Information flow:

- After f, we can recover $b_1 + b_3$ and $b_2 + b_3$
- We do not know $\mathrm{b}_1 + b_2 - b_3$ anymore
  - Anything along that line, i.e. $(\lambda, \lambda, -\lambda), \lambda \in \mathbb{R}$

Not invertible

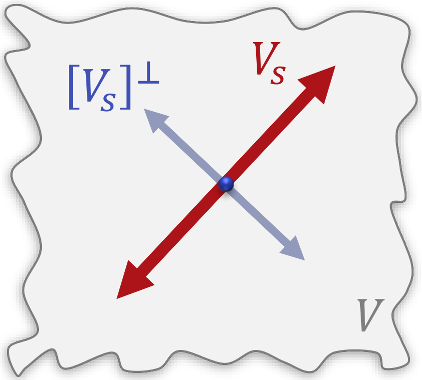


$$f(\mathbf{x}) = \begin{pmatrix} x_1 + x_3 \\ x_2 + x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \mathbf{x}$$

Information flow:

- After f, we can recover $b_1 + b_3$ and $b_2 + b_3$
- We do not know $b_2 - b_3$ anymore

Orthogonal Comlement



Definition

- **Given:** Subspace $V_s \subseteq V$
- Orthogonal complement

$$V_s^\perp := \{\mathbf{v} \in V \,|\, \forall \mathbf{w} \in V_s : \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$$

Intuition

- Set of all vectors orthogonal to V_s
- Zero projection onto any $\mathbf{w} \in V_s$

Theorem

$$V_s \subset V \Rightarrow V = \text{span}\{V_s, V_s^\perp\} \quad [:= V_s \oplus V_s^\perp]$$

In general

Consider mapping

$$f: V_1 \to V_2$$

Subspaces of V_1

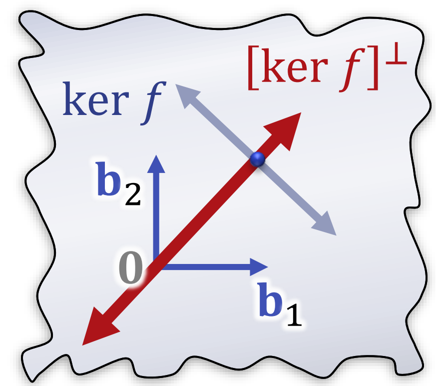
- **Kernel:** Subspace that is lost

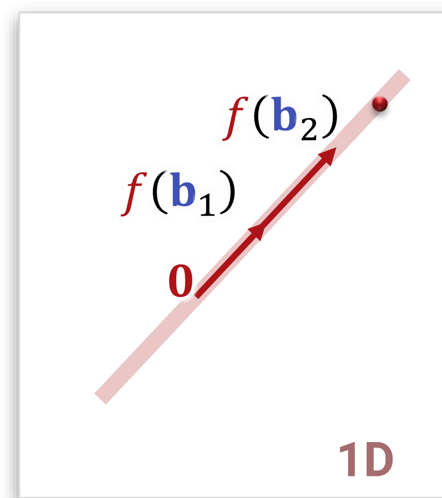
$$\ker f := \{\mathbf{x} \in V_1 | f(\mathbf{x}) = 0\}$$

- **Orthogonal complement of kernel**

$$[\ker f]^{\perp} = \{\mathbf{v} \in V_1 | \forall \mathbf{w} \in \ker f : \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$$

  - In this space, f is invertible





In general

Consider mapping

$$f : V_1 \rightarrow V_2$$

In the target domain

$$\operatorname{im} f := \{\mathbf{y} \in V_2 | \exists \mathbf{x} \in V_1 : f(\mathbf{x}) = \mathbf{y}\}$$

- Subspace of V_2
- Same dimension as kernel complement

$$\dim([\ker f]^{\perp}) = \dim(\operatorname{im} f)$$

In general

Consider mapping

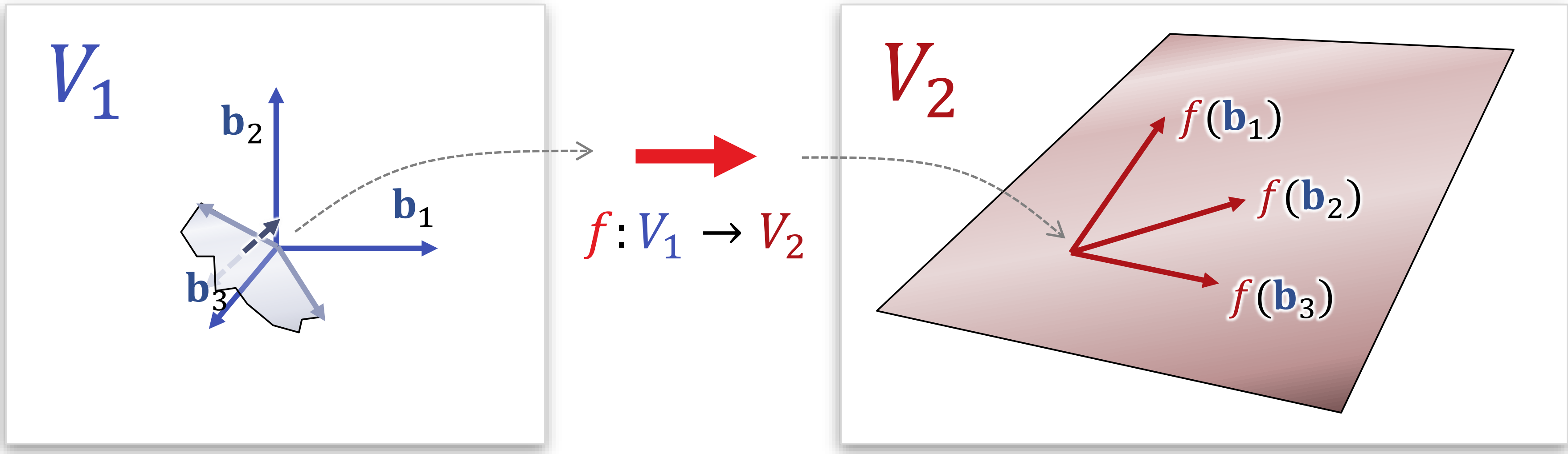
- Rank is the dimension of the mapped space

$$\mathrm{rank}(f) := \dim(\mathrm{im}\, f)$$
$$= \dim\big(\mathrm{span}(V_1 \setminus \ker f)\big)$$

- Source space V_1 is split:
  - $\dim \mathrm{im}(f)$ = dimensions "preserved" by f
  - $\dim \ker(f)$ = dimensions "removed" by f
- Sums up:

$$\dim(V_1) = \dim(\mathrm{im}\, f) + \dim(\ker f)$$

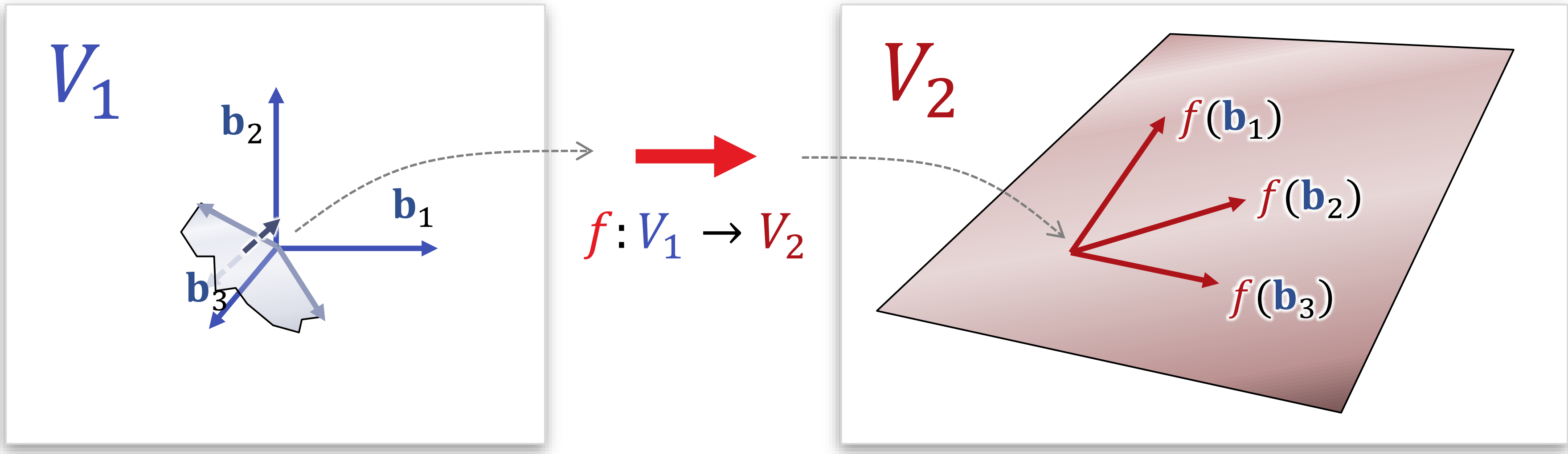
Structural Insight



Mapping Subspaces to Subspaces

- Invertible map from $[\ker f]^{\perp} \to \operatorname{im} f$
- Not covered
  - “Source” information lost: coordinates within $\ker f$
  - Unreachable ”targets”: vectors within $[\operatorname{im} f]^{\perp}$

Structural Insight



Dimensions add up

- $\dim [\ker f]^{\perp} = \dim \operatorname{im} f$
- $\dim V_1 = \dim \ker f + \dim [\ker f]^{\perp}$
- $\dim V_2 = \dim \operatorname{im} f + \dim [\operatorname{im} f]^{\perp}$

In practice?

In practice

- It always never works:
  - Most matrices have noise (measurement, numerics)
    - Any practical mapping has "full rank"
  - Inverting matrices is not always stable
    - Even full-rank matrices might delete information
  - Need to understand this better!

We will discuss this soon

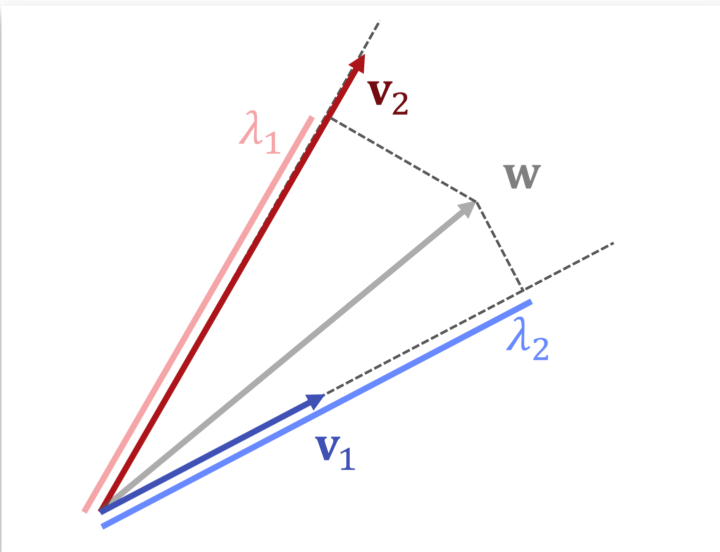
- Tools:
  - Eigenvalues
  - Singular value decomposition (SVD)

Linear Systems of Equations

Inverting Linear Maps

Situation

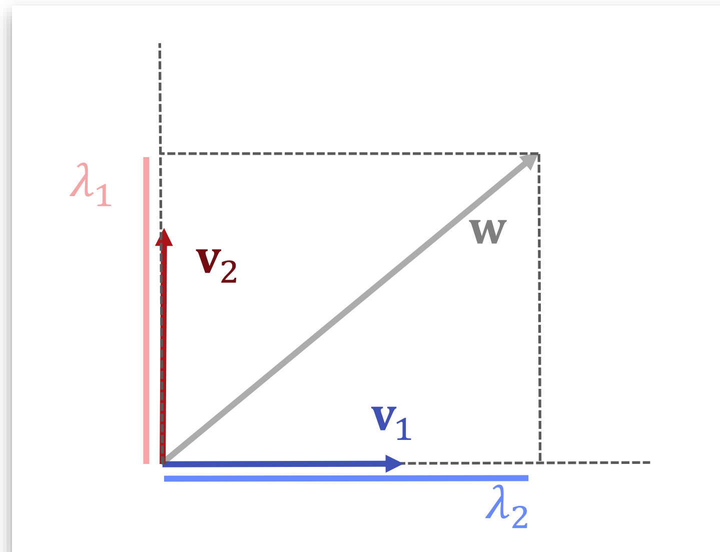
General Case



Linear System

$$\lambda_1 \cdot \mathbf{v}_1 + \cdots + \lambda_n \cdot \mathbf{v}_n = \mathbf{w}$$

Orthogonal



Direct Computation

$$\lambda_1 = \mathbf{v}_1 \cdot \mathbf{w}$$

$$\vdots$$

$$\lambda_n = \mathbf{v}_n \cdot \mathbf{w}$$

Linear Systems of Equations

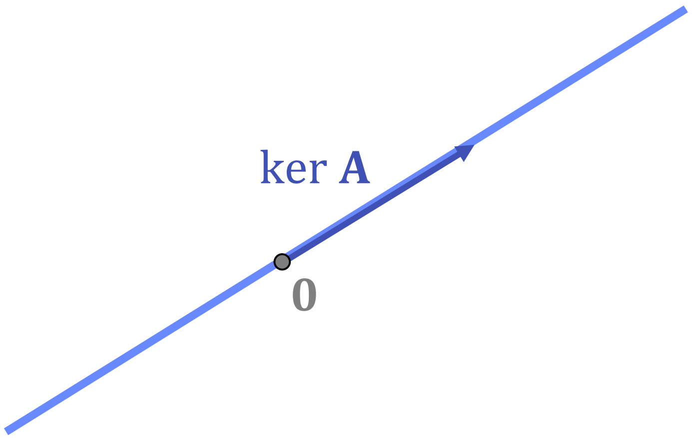
Problem: Invert an affine map

- Given: $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, i.e, $\mathbf{A} \cdot \mathbf{x} - \mathbf{b} = \mathbf{0}$
  - We know $\mathbf{A}$, $\mathbf{b}$
  - Looking for $\mathbf{x}$
- Compute $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$

Solution

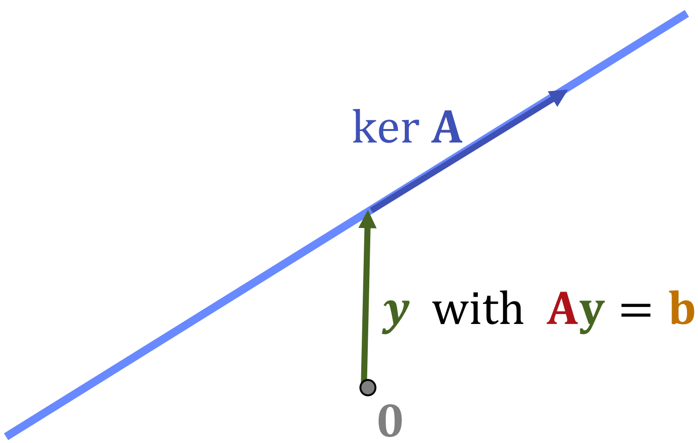
- Set of solutions: *affine subspace* of $\mathbb{R}^n$ (or $\emptyset$)
  - Point, line, plane, hyperplane...
- Innumerous algorithms

Linear Systems of Equations



$\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$ – hyperplane through the origin

"Homogeneous" system



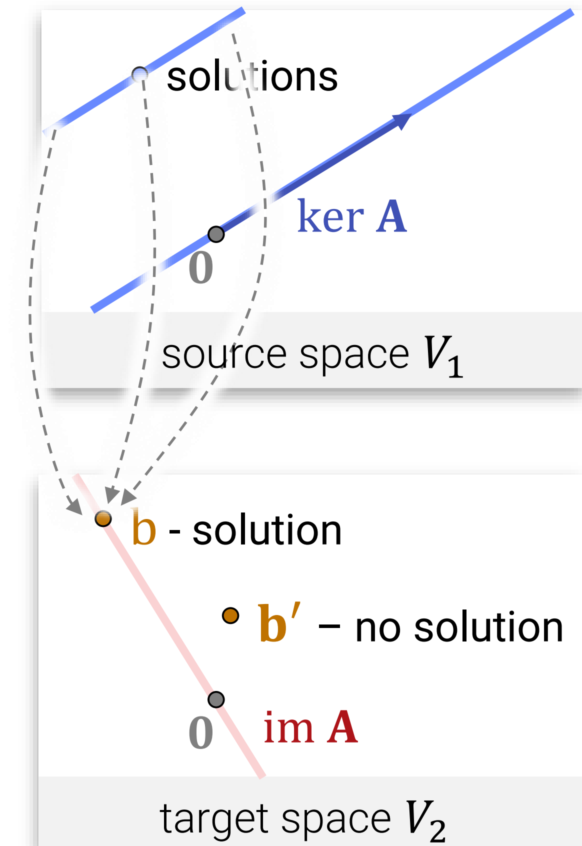
$\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ – hyperplane through any point

"Inhomogeneous" system

Structure

Linear System ($\mathbf{A}: V_1 \to V_2$):

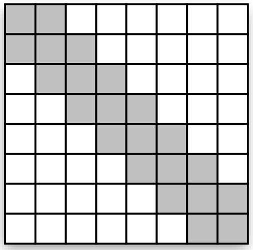
- $\mathbf{A}\mathbf{x} = \mathbf{0}$
  - Solution space = $\ker \mathbf{A}$
- $\mathbf{A}\mathbf{x} = \mathbf{b}$
  - Might or might not have a solution
  - Solution if and only if $\mathbf{b} \in \operatorname{im} \mathbf{A}$
- Set of all solutions:
  - One $\mathbf{y}$ with $\mathbf{A}\mathbf{y} = \mathbf{b}$
  - Add any solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$
  - Solution set: $\mathbf{y} + \ker \mathbf{A}$



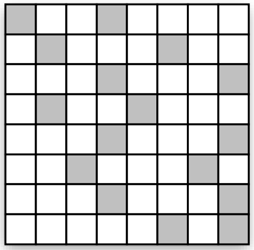
Solvers for Linear Systems

Solving linear systems of equations

- **Baseline:** Gaussian elimination
  $O(n^3)$ operations for $n \times n$ matrices
- We can do better, in particular for special cases:
  - **Band matrices:**
    constant bandwidth



  - **Sparse matrices:**
    constant number of non-zero entries per row
    – Store only non-zero entries



Solvers for Linear Systems

Algorithms: linear systems of *n* equations

- Band matrices, O(1) bandwidth:
  - Modified O(n) elimination algorithm.
- Iterative Gauss-Seidel solver
  - converges for diagonally dominant matrices
  - Typically: O(n) iterations, each costs O(n) for a sparse matrix.
- Conjugate Gradient solver
  - Only symmetric, positive definite matrices
  - Guaranteed: O(n) iterations
  - Typically good solution after O($\sqrt{n}$) iterations.

See: J. R. Shewchuk, An Introduction to the Conjugate Gradient Method Without the Agonizing Pain, 1994.